End-Semester Exam ISI Bangalore 6th May 2013

Representation Theory of finite groups 100 Points

Notes.

(a) Leave sufficient gap between your answers to different questions. Preferably, begin each answer on a separate page.

(b) Justify all your steps. Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(d) Generic notation: G is a group, k is a field, V is a G-module over k. Unless specified otherwise, V will be assumed to be finite-dimensional.

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0. [4 points] These points are for neatness and clarity of presentation.

1. [16 points] Write down the character table for the permutation group S_4. Briefly describe the irreducible representation corresponding to each character.

2. [16 points] Let $\mathbb{V}$ denote the standard module for the permutation group S_3. Express $\mathbb{V}^{\otimes 3}$ as a direct sum of irreducible S_3-modules.

3. [16 points] Let G be a finite group and V a G-module over $\mathbb{C}$. Prove that

$$3\chi_{\bigwedge^3(V)}(g) = \frac{1}{2}\chi_V(g)^3 - \frac{3}{2}\chi_V(g^2)\chi_V(g) + \chi_V(g^3)$$

where χ_- denotes character of a representation.

4. [16 points] Let $G = C_3 = \{1, \tau, \tau^2 \mid \tau^3 = 1\}$ be the cyclic group of order 3. Consider the $\mathbb{C}$-linear action of G on the space V of polynomials of at most degree 1 in $\mathbb{C}[X, Y, Z]$ via

$$\tau \cdot X = Y, \quad \tau \cdot Y = Z, \quad \tau \cdot Z = X.$$

Decompose V as a sum of irreducible G-modules.

5. [16 points] For $i = 1, 2$, let G_i be finite groups and let V_i be irreducible G_i-modules. Prove that $V_1 \oplus V_2$ has a natural structure of $G_1 \times G_2$-module for which it is irreducible.

6. [16 points] Let $G = A_4$ be the alternating group on 4 symbols. List the 4 projection operators in $\mathbb{C}[G]$ corresponding to the 4 irreducible G-modules. You may find it useful to allot names to the sum of all elements in a conjugacy class, e.g., say, $a = (12)(34) + (13)(24) + (14)(23)$.